Topology I

Fourth Year

First Semester

Theorem:

Let (X, τ) be a topological space and $A, B \subseteq X$, then:

1- $bd(A) \cap Int(A) = \emptyset$, $bd(A) \cap Ext(A) = \emptyset$.

2- $bd(A) = bd(A^c)$.

3- $bd(A \cup B) \subseteq bd(A) \cup bd(B)$.

4- $A \in \tau$, i.e., A is open iff $bd(A) \subseteq A^c$ and $bd(A) \cap A = \emptyset$.

5- $A^c \in \tau$, i.e., A^c is open iff $bd(A) \subseteq A$ and $bd(A) \cap A^c = \emptyset$.

6- $A, A^c \in \tau$, i.e., A and A^c are open iff $bd(A) = \emptyset$

Topology I

Recall some basic concepts of set theory which are stated some of them next as a basic concepts.

1- The classical set of objects X is called the universal set whose generic elements are denoted by x, and if x belongs to X (written as $x \in X$) while if x does not belongs to X (written as $x \notin X$).

2- The empty set $\emptyset$ has no elements and so $x \notin \emptyset$, for all $x \in X$ and when a set A subset A of X (written as $A \subseteq X$) and when $A \subset X$ it is called a proper subset if A contains at least one element of X, then it is nonempty.

3- Let A and B be two subsets of X, then we say that A is a subset of B (written $A \subseteq B$) iff $x \in A \rightarrow x \in B$.

4- The two sets A and B are said to be equal (written as $A = B$) if $A \subseteq B$ and $B \subseteq A$.

5- Let A and B be sets. The intersection $A \cap B$ is the set of all objects that belongs to A and B, i.e.,

$$A \cap B = \{x \in X : x \in A \text{ and } x \in B\}$$

and it is clear that $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

6- The union $A \cup B$ is the set all objects that are elements of A or B or both, i.e.,

$$A \cup B = \{x \in X : x \in A \text{ or } x \in B\}$$

and it is clear also that $A \subseteq A \cup B$, $B \subseteq A \cup B$

7- $A \cup B = B \cup A$, $A \cap B = B \cap A$, $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$,

8- Let $A \subseteq X$, then the complement of A with respect to X is denoted by A^c (or $X - A$ or $X \setminus A$), s.t $A^c = \{x \in X : x \notin A\}$.

9- DeMorgan's laws

$$(A \cup B)^c = A^c \cap B^c, \quad (A \cap B)^c = A^c \cup B^c$$

10- For a given set X, the collection of all subsets $A \subset X$ is

called the power set of A and is denoted by $P(A)$, i-e, $P(X) = \{A : A \subseteq X\}$.

11- The set of natural numbers $\mathbb{N}$ is countable set.

12- A set A is said to be countable if there is a 1-1 correspondence between A and the set of natural numbers $\mathbb{N}$. Otherwise it is uncountable. (Countable sets may be finite or infinite).

13- A function f from a set A to a set B is a rule that to each element $x \in A$ associates a unique element $f(x) \in B$ (A is called the domain of f and B the range or codomain of f), where the function f is indicated like $f: A \to B$.

14- A function $f: A \to B$ is said to be injective (or one-to-one, i-e, 1-1) if $f(x) = f(y)$ only if $x = y$, $\forall x, y \in A$. We say that f is surjective (or onto) if for each $y \in B$, there exist $x \in A$ such that $f(x) = y$. We say that f is bijective (or one-to-one correspondence) if it is both injective and surjective (i-e, if 1-1 and onto)

15- When $f: A \to B$ is bijective (i-e, if 1-1 and onto), then f has an inverse $f^{-1}: B \to A$, f^{-1} is called the inverse function of f.

16- The Cartesian Product of sets $A_1, A_2, \ldots, A_n$ is defined by $A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i, \forall i = 1, \ldots, n\}$

17- Two sets A and B are said to be disjoint if $A \cap B = \emptyset$.

18- If X is the universal set and $A \subset X$, then $A \cap A^c = \emptyset$ and $A \cup A^c = X$.

19- Each of the following conditions is equivalent to $A \subseteq B$
(i) $A \cap B = A$, (ii) $A \cup B = B$, (iii) $B^c \subseteq A^c$, (iv) $B \cup A^c = X$, where X is the universal set, (v) $A \cap B^c = \emptyset$.

The Cofinite Topology:

Definition:

Let X be a nonempty set, the cofinite topology $T_{cof.}$ is the collection of all subsets $U \subset X$ whose complement $X-U$ is finite together with the empty set, i.e.

$$T_{cof.} = \{U \subset X \mid X-U \text{ is finite}\} \cup \{\emptyset\}$$

Lemma:

The collection $T_{cof.}$ is a topology on X.

Proof:

We must prove that the three condition of a topology are satisfied on $T_{cof.}$.

1- By definition $\emptyset \in T_{cof.}$ and and also $X \in T_{cof.}$ since $X-X = \emptyset \in T_{cof.}$

2- Let $\{U_i\}_{i \in I}$ be a subcollection of open sets which belongs to $T_{cof.}$, i.e., let $U_i \in T_{cof.}$, $\forall i \in I$ T.P. $\bigcup_{i \in I} U_i \in T_{cof.}$

∵ For each $i \in I$, $U_i \in T_{cof.} \rightarrow X-U_i$ is finite or $\emptyset$

Let $V = \bigcup_{i \in I} U_i$ and T.P. $X-V$ is finite or $\emptyset$

If each U_i is empty $\rightarrow V$ is empty $\rightarrow V \in T_{cof.}$

Otherwise, take any $n \in I$ such that $U_n \subset V$

$\rightarrow X-U_n$ is finite

∵ $U_n \subset V \rightarrow X-V \subset X-U_n$ which is finite

∵ $X-V$ is a subset of a finite set, which is also finite $\rightarrow X-V$ is finite $\rightarrow \bigcup_{i \in I} U_i \in T_{cof.}$

3- Let $U_1, U_2, \dots, U_n \in T_{cof.}$

$\rightarrow X-U_1, X-U_2, \dots, X-U_n$ are finite or $U_i = \emptyset$,

Let $W = U_1 \cap U_2 \cap \dots \cap U_n$

T.P. $X-W$ is finite or $\emptyset$

If each U_i is empty $\rightarrow W \subset U_i$, $\forall i=1,2,\dots,n$ is empty
otherwise, using DeMorgan's law,

$$X - W = X - (U_1 \cap U_2 \cap \dots \cap U_n)$$
$$= (X - U_1) \cup (X - U_2) \cup \dots \cup (X - U_n)$$

where each $X - U_i$, $\forall i=1,2,\dots,n$ is finite
Since the finite union of finite sets is finite
$\therefore$ $X - W$ is finite $\rightarrow W \in T_{cof.}$, i.e., $\bigcap_{i=1}^{n} U_i \in T_{cof}$
$\therefore$ $T_{cof.}$ is a topology on X. #

The usual Topology:

Definition:

Let $X = \mathbb{R}$ (set of real numbers), the usual topology T_u is the collection of all open intervals $(a,b) \subset \mathbb{R}$, with the empty set and $\mathbb{R}$, i.e.,

$$T_u = \{U = (a,b) \subset \mathbb{R} \mid \forall x \in U, \exists (a,b) \subset \mathbb{R} \text{ such that } x \in (a,b) \subseteq U\} \cup \{\emptyset\} \cup \mathbb{R}$$

or $T_u = \{U \subseteq \mathbb{R} \mid U$ is a union ~~of~~ a family of open intervals (a,b) in $\mathbb{R}\}$

Lemma:

The collection T_u is a topology on $\mathbb{R}$, and $(\mathbb{R}, T_u)$ is called the usual topological space

proof:

1- From definition $\mathbb{R} = (-\infty, \infty) \in T_u$ is considered as an open interval and also $\emptyset \in T_u$

بعض الكتب والمصادر تعتبر $\emptyset = (a,a) \in T_u$

2- Let $U_i \in T_u$, $i \in I$
If for some i, $U_i = \mathbb{R}$, then $\bigcup_{i \in I} U_i = \mathbb{R} \in T_u$
If for all i, $U_i = \emptyset$, then $\bigcup_{i \in I} U_i = \emptyset \in T_u$
Now, let $x \in \bigcup_{i \in I} U_i \rightarrow x \in U_i$, for some i
$\rightarrow \exists (a,b) \subset \mathbb{R}$ s.t $x \in (a,b) \subseteq U_i$
$\rightarrow x \in (a,b) \subseteq U_i \subseteq \bigcup_{i \in I} U_i$
$\rightarrow x \in (a,b) \subseteq \bigcup_{i \in I} U_i$
$\therefore \bigcup_{i \in I} U_i \in T_u$

3- Let $U, V \in T_u$

If U or V or both equals $\emptyset$, then $U \cap V = \emptyset \in T_u$

If $U = \mathbb{R}$, then $U \cap V = \mathbb{R} \cap V = V \in T_u$

also if $V = \mathbb{R}$, then $U \cap V = U \cap \mathbb{R} = U \in T_u$

otherwise, let $x \in U \cap V \rightarrow x \in U \wedge x \in V$

3° $x \in U \rightarrow \exists$ an open interval (a,b) s.t $x \in (a,b) \subseteq U$

$x \in V \rightarrow \exists$ an open interval (c,d) s.t $x \in (c,d) \subseteq V$

°° $x \in (a,b)$ and $x \in (c,d)$, such that

$x \in (a,b) \cap (c,d) \subseteq U \cap V$

$\rightarrow x \in (\max\{a,c\}, \min\{b,d\}) \subseteq U \cap V$

°° $\exists$ an open interval $(\max\{a,c\}, \min\{b,d\})$ such that $x \in (\max\{a,c\}, \min\{b,d\}) \subseteq U \cap V$

°° $U \cap V \in T_u$

Hence for any finite sets $U_1, U_2, \dots, U_n \in T_u$, we have $U_1 \cap U_2 \cap \dots \cap U_n \in T_u$

3° T_u is a topology on $\mathbb{R}$ and $(\mathbb{R}, T_u)$ is a topological space which is called the usual topological space #

Theorem:

Let (X, T) be a topological space and $\mathcal{F}$ be a family of closed sets on X, then:

1- $X, \emptyset \in \mathcal{F}$.

2- If $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$.

3- If $A_i \in \mathcal{F}, \forall i \in I$, then $\bigcap_{i \in I} A_i \in \mathcal{F}$.

Proof:

1- °° $X \in T \rightarrow X^c = \emptyset \in \mathcal{F}$ (since the complement of open set is closed

$\emptyset \in T$ (open) $\rightarrow \emptyset^c = X \in \mathcal{F}$

2- Let $A, B \in \mathcal{F}$ (A, B are closed sets)

$\rightarrow A^c, B^c \in T$ (A^c, B^c are open sets)

$\rightarrow A^c \cap B^c \in T$ (By the def. of topology)

$\rightarrow (A \cup B)^c \in T$ (By DeMorgan's law

$\rightarrow A \cup B \in \mathcal{F}$ (since by the def. of closed sets

3. Let $A_i \in \mathcal{F}$, $\forall i \in I$ (each A_i is closed)

$\rightarrow A_i^c \in \tau$, $\forall i \in I$

$\rightarrow \bigcup_{i \in I} A_i^c \in \tau$ (by condition 2 of topology)

$\rightarrow (\bigcap_{i \in I} A_i)^c \in \tau$ (By Demorgan's law)

i.e., $(\bigcap_{i \in I} A_i)^c$ is open $\rightarrow \bigcap_{i \in I} A_i$ is closed, i.e.,

$\bigcap_{i \in I} A_i \in \mathcal{F}$. #

Remarks:

Let (X, τ) be a topological space, then a set A of X may be

1- open but not closed.
2- closed but not open.
3- closed and open in the same time (called clopen).
4- Not open and not closed.

Remarks:

Consider the usual topological space $(\mathbb{R}, \tau_u)$, Then:

1- The sets $(0,1) \cup (2,4)$, $(-2,1)$, ..., i.e., any interval of the form (a,b), $a,b \in \mathbb{R}$ are open sets in τ_u.

2- The set of natural numbers $\mathbb{N}$ is not an open set in τ_u, since it can not be represented as a union of open intervals, but it is closed since

$\mathbb{N}^c = \{1, 2, 3, \ldots\}$

$= \underbrace{(-\infty, 1)}_{U_1} \cup \underbrace{(1,2)}_{U_2} \cup \underbrace{(2,3)}_{U_3} \cup \ldots = \bigcup_{i=1}^{\infty} U_i \in \tau_u$

3- From (2) every set containing a discrete set of points, say $\{x_1, x_2, \dots, x_n\}$ is a closed set in T_u.

4- The set $[a,b]$ is a closed interval in T_u, since $[a,b]^c = (-\infty, a) \cup (b, \infty) \in T_u$ (i.e., is open) $\Rightarrow$ $[a,b]$ is closed interval in T_u or a closed set in T_u

5- The set of rational numbers $\mathbb{Q}$ and the set of irrational numbers $\mathbb{I}$ are not open and not closed sets in T_u, since from the property that (between any two rational numbers there exist an irrational number) and hence every open set (a,b) of $\mathbb{R}$ contains rational and irrational numbers

Examples:

Let $X = \mathbb{R}$ and define $T = \{\mathbb{R}, \emptyset, \mathbb{Q}, \mathbb{I}\}$, then T is a topology on $\mathbb{R}$ (different from the usual topology) since

1- $\emptyset, \mathbb{R} \in T$

2- $\mathbb{Q}, \mathbb{I} \in T \rightarrow \mathbb{Q} \cup \mathbb{I} = \mathbb{R} \in T$ and similarly for any open sets A_i in T, we have $\bigcup A_i \in T$

3- $\mathbb{Q}, \mathbb{I} \in T \rightarrow \mathbb{Q} \cap \mathbb{I} = \emptyset \in T$

The open intervals in T_u are not open in T except $\mathbb{R}$ and $\emptyset$, while the open sets in T are $\mathbb{R}, \emptyset, \mathbb{Q}, \mathbb{I}$ which are closed also in T.

Remarks

Let X be any nonempty set and A be a proper subset of X, then:

1- $T = \{X, \emptyset, A\}$ is a topology on X.

2- $T = \{X, \emptyset, A, A^c\}$ is a topology on X and this topology has the property that every set in T is clopen and hence $T = P$.

Example(1)

Let X be a nonempty set and let $x_0 \in X$. Define

$$T = \{U \subseteq X : x_0 \in U\} \cup \{\emptyset\}$$

Show that (X, T) is a topological space.

Solution

1- $\emptyset \in T$ be the definition of T

$X \in T$ since X is the universal set which contains all elements and so among them is $x_0 \in X$

2- Let $U_i \in T$, $\forall i \in I$

If $U_i = \emptyset$, $\forall i \longrightarrow \bigcup_{i \in I} U_i = \emptyset \in T$

If $U_i \neq \emptyset$, for some $i \longrightarrow x_0 \in U_i$, for some i

$\longrightarrow x_0 \in \bigcup_{i \in I} U_i \longrightarrow \bigcup_{i \in I} U_i \in T$

3- Let $U, V \in T$

If $U = \emptyset$ or $V = \emptyset$, then $U \cap V = \emptyset \in T$

If $U = X$, then $U \cap V = V \in T$

If $V = X$, then $U \cap V = U \in T$

If $U, V \neq \emptyset$ or X, then

$x_0 \in U \wedge x_0 \in V$

$\longrightarrow x_0 \in U \cap V$

$\longrightarrow U \cap V \in T$

$\therefore (X, T)$ is a topological space.

Example(2) (H.w.)

Let X be a nonempty set and let $x_0 \in X$. Define

$$T = \{U \subseteq X : x_0 \notin U\} \cup \{X\}$$

Show that (X, T) is a topological space.

Remarks on Example (1):

1- In example (1), any set which contains x_0 is open (by def.), and hence as a special case when $X = \mathbb{R}$ and $x_0 = 0$, then the sets $\{0\}, (-\infty, 2), \mathbb{Q}, \{0, 1\}, \ldots$ are open sets. while $\{-4\}, \mathbb{I}, \mathbb{N}, (6, \infty), [3, 5], \ldots$

are closed sets since they do not contains 0

2- In general, we can replace 0 by 2 or $\sqrt{5}$ or any other real number and we can replace $\mathbb{R}$ by any other set to find the open and closed sets

Remarks on Example (2):

A special case of this example when we take $X=\mathbb{N}$ and $x_0=2$, then the open sets are $\mathbb{N}$ and every subset of $\mathbb{N}$ which do not contain 2, say $\{1\}, \{3,4,\dots\}, \{3,4\}, \dots$ are open, while $\{2,3,\dots\}, \{1,2\}, \{1,2,5,6,\dots\}$ are closed with respect to this topology

Example:

Let $X=\mathbb{N}$ and define $\tau=\{A_n\subseteq\mathbb{N}: A_n=\{1,2,\dots,n\}, n\in\mathbb{N}\} \cup\{\mathbb{N}\}\cup\{\emptyset\}$

Show that $(\mathbb{N},\tau)$ is a topology.

Solution

We can note that the elements or subsets of τ are $A_1=\{1\}, A_2=\{1,2\}, A_3=\{1,2,3\},\dots$

$\longrightarrow A_1\subseteq A_2\subseteq A_2\subseteq A_3\subseteq\dots$

Now:

1- $\mathbb{N}, \emptyset\in\tau$ by definition

2- Let $A_i\in\tau, i\in I$, then $\bigcup_{i\in I} A_i=\begin{cases} A_k, \text{ if } i \text{ is finite and } k\geq i, \forall i \\ \mathbb{N}, \text{ if } i \text{ is infinite} \end{cases}$ which belongs to τ in both cases

3- Let $A_i, A_j\in\tau$, then

$$A_i\cap A_j=\begin{cases} A_i\in\tau, & \text{if } i\leq j \\ A_j\in\tau, & \text{if } i>j \end{cases}$$

3. $(\mathbb{N},\tau)$ is a topological space.

As an illustration of this example $A_1=\{1\}, A_{10}=\{1,2,\dots,10\}, \mathbb{N}, \emptyset,\dots$ are among open sets in τ and $A_1^c=\{2,3,\dots\}, A_{10}^c=\{11,12,\dots\}, \emptyset, \mathbb{N},\dots$ are among closed sets in τ

H.W:
Let $X = \mathbb{N}$ and $T = \{B_n \subseteq \mathbb{N} : B_n = \{n, n+1, \dots\}, n \in \mathbb{N}\} \cup \{\emptyset\}$
Show that T is a topology on $\mathbb{N}$.

Theorem:
Let $\{T_i\}, i \in I$ be a family of topologies on X, then
1- $\bigcap_{i \in I} T_i$ is a topology on X.
2- The union of topologies on X need not to be a topology on X.

Proof:
1- Let $\{T_i\}_{i \in I}$ be the family of topologies on X and let $T = \bigcap_{i \in I} T_i$
T.P. T is a topology on X
(i) Since each of $T_i, \forall i$ is a topology, then
$\emptyset, X \in T_i, \forall i \rightarrow \emptyset, X \in \bigcap_{i \in I} T_i = T$
(ii) Let $U_j \in T = \bigcap_{i \in I} T_i \rightarrow U_j \in T_i, \forall i, j$
and since each of T_i is a topology, then
$\bigcup_{j \in J} U_j \in T_i, i \in I$
$\rightarrow \bigcup_{j \in J} U_j \in \bigcap_{i \in I} T_i = T \rightarrow \bigcup_{j \in J} U_j \in T$
(iii) Let $U_1, U_2 \in T = \bigcap_{i \in I} T_i$
$\rightarrow U_1, U_2 \in T_i, \forall i$
and since $T_i, \forall i$ is a topology, then
$U_1 \cap U_2 \in T_i, \forall i$
$\rightarrow U_1 \cap U_2 \in \bigcap_{i \in I} T_i = T$
$\therefore (X, T) = (X, \bigcap_{i \in I} T_i)$ is a topological space
2- Let $X = \{a, b, c\}$ and consider two topologies on X, namely
$T_1 = \{\emptyset, X, \{a\}\}, T_2 = \{\emptyset, X, \{b\}\}$
consider $T = T_1 \cup T_2 = \{\emptyset, X, \{a\}, \{b\}\}$
Then $T = T_1 \cup T_2$ is not a topology on X
Since $\{a\}, \{b\} \in T$, but $\{a\} \cup \{b\} = \{a, b\} \notin T$.
#

Definition:

Let τ_1 and τ_2 be two topologies on a given non empty set X, then we say that

1- τ_2 is finer (larger) than τ_1 or is coarser (smaller) than than τ_2 if $\tau_1 \subset \tau_2$.

2- τ_1 is comparable with τ_2 if either $\tau_1 \subset \tau_2$ or $\tau_2 \subset \tau_1$.

Remark: (indiscrete)

The trivial topology is coarser than any other topology and the discrete topology is finer than any other topology.

Proof:

Let τ be any topology on X, then we have

$\tau_{triv.} = \{\emptyset, X\} \subset \tau \subset P(X) = \tau_{disc.}$ #

Examples:

Let $X = \mathbb{N}$, then the discrete topology $\tau_{disc.}$ on $\mathbb{N}$ is finer than the cofinite topology $\tau_{cof.}$ on $\mathbb{N}$, which is finer than the trivial topology $\tau_{triv.}$ on $\mathbb{N}$.

Example:

Let $X = \mathbb{R}$ and consider the usual topology τ_u and the cofinite topology. Then τ_u and $\tau_{cof.}$ are comparable topologies on $\mathbb{R}$, since $\tau_{cof.} \subset \tau_u$, because

Solution:

Let $O \in \tau_{cof.}$, then either $O = \emptyset$ or O^c is finite

If $O = \emptyset \longrightarrow \emptyset \in \tau_u$

If O^c is finite, then $O^c = \{x_1, x_2, \dots, x_n\}$ is closed. $\Rightarrow O$ is open

where $x_1, x_2, \dots, x_n \in \mathbb{R}$ and indeed

$O = (-\infty, x_1) \cup (x_1, x_2) \cup \dots \cup (x_{n-1}, x_n) \cup (x_n, \infty)$

which is open in τ_u since the union of open

subintervals of $\mathbb{R}$ with respect to the topology T_u is open
$\Rightarrow O \in T_u$
Hence, $T_{cof} \subset T_u$
Therefore $T_{cof.}$ and T_u are comparable and $T_{cof.}$ is the is the coarser topology while T_u is the finer topology
In addition $T_{cof.} \neq T_u$, since the open interval $(5,7) \in T_u$, but $(5,7) \in T_{cof.}$

Remark:
If X is a nonempty set, then T is the discrete topology on X if and only if $\{x\} \in T$, $\forall x \in X$ (i.e., T contains all singelto elements of X at least in addition to $\emptyset$ and X).

Proof:
$\Rightarrow$ If T is the discrete topology, then $T = P(X)$ (the Power Set of X)
and since among the subsets of the power set are those sets with single elements, then $\forall x \in X$ we have $\{x\} \in T$.

$\Leftarrow$ Let $\{x\} \in T$, $\forall x \in X$ and T.P. T is the discrete topology
It is enough to prove $P(X) \subseteq T$
If $A \in P(X) \rightarrow A = \bigcup_{x \in A} \{x\} \in T$ (by condition (2) of the topological spaces)
$\Rightarrow P(X) \subseteq T$ and hence $T = P(X)$, i.e., T is the discrete topology on X.

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Theorem:
Let X and Y be a nonempty sets, and (Y, T_Y) is a topological space. If $f: X \rightarrow Y$ is a mapping, then $T_X = \{f^{-1}(B) : B \in T_Y\}$ is a topology on X, where f^{-1} is the inverse image.

Proofs:

1- Since $\phi = f^{-1}(\phi)$ and $\phi \in T_Y$, then $\phi \in T_X$
$X = f^{-1}(Y)$ and $Y \in T_Y$, then $X \in T_X$.

2- Let $\{U_i\}_{i\in I} \in T_X$, T.P. $\bigcup_{i\in I} U_i \in T_X \rightarrow \exists B_i \in T_Y, \forall$
$\because U_i = f^{-1}(B_i)$, $B_i \in T_Y$, $\forall i \in I$
$\rightarrow \bigcup_{i\in I} U_i = \bigcup_{i\in I} f^{-1}(B_i) = f^{-1}(\bigcup_{i\in I} B_i) = f^{-1}(B)$, where
f^{-1} preserve the union and $B = \bigcup_{i\in I} B_i \in T_Y$ since T_Y is a topology
$\therefore$ By def. of T_X, we have $\bigcup U_i \in T_X$

3- Let $U_1, U_2 \in T_X$, T.P. $U_1 \cap U_2 \in T_X$
$U_1 \in T_X \rightarrow \exists B_1 \in T_Y$ such that $U_1 = f^{-1}(B_1)$
$U_1 \in T_X \rightarrow \exists B_2 \in T_Y$ such that $U_2 = f^{-1}(B_2)$
Also, the induced mapping f^{-1} preserve intersection
$\therefore U_1 \cap U_2 = f^{-1}(B_1) \cap f^{-1}(B_2)$
$= f^{-1}(B_1 \cap B_2)$
$= f^{-1}(B)$, $B = B_1 \cap B_2 \in T_Y$ since T_Y is a Top.
Therefore $U_1 \cap U_2 \in T_X$.
$\therefore T_X$ is a topology on X and (X, T_X) is also a topological space.

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Definition:
Let (X, T_X) and (Y, T_Y) be two topological spaces, then (X, T_X) and (Y, T_Y) are said to be equal topologies and is denoted by $(X, T_X) = (Y, T_Y)$ if $X = Y$ and $T_X = T_Y$.

In a general topological spaces, we cannot speak about balls around points, because there is no notion concerning of distances between points or elements.

Definition:
Let (X, T) be a topological space and $x \in X$. A point $P \in X$ is an accumulation point or limit point

or cluster point or derived point of a subset A of X iff every open set G containing p contains a point of A different from p, i.e., $\forall\, G \in \mathcal{T}$, such that $p \in G$, implies

$$(G - \{p\}) \cap A \neq \phi$$

The set of all accumulation points of A is denoted by A' (called the derived set of A).

Examples

Let $X = \{a, b, c, d, e\}$ and define a topology on X by $\mathcal{T} = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$

Take $A = \{a, b, c\} \subset X$ and to find the limit or accumulation points of A

Take $a \in X$, then the open sets containing a are

$$X, \{a\}, \{a, c, d\}, \text{ i.e., } a \in X, \{a\}, \{a, c, d\}$$

Since $(\{a\} - \{a\}) \cap A = \phi \cap A = \phi$

$\therefore$ a is not a limit point

$b \in X \longrightarrow b \in \{b, c, d, e\}, X$ (open sets)

$\rightarrow (\{b, c, d, e\} - \{b\}) \cap A = \{c, d, e\} \cap \{a, b, c\}$

$= \{c\} \neq \phi$

$(X - \{b\}) \cap A = \{a, c, d, e\} \cap \{a, b, c\} = \{a, c\} \neq \phi$

$\therefore$ b is a limit point

$c \in X \longrightarrow c \in \{c, d\}, \{a, c, d\}, \{b, c, d, e\}, X$ (open sets)

$(\{c, d\} - \{c\}) \cap A = \{d\} \cap \{a, b, c\} = \phi$

$\therefore$ c is not a limit point

Similarly $d, e \in X$ and they are limit points of A (by definition) check.

Hence, $A' = \{b, d, e\}$ is the derived set of A.

Example 3

Let X be any nonempty set and $\mathcal{T}$ is the trivial or discrete topology on X, i.e., $\mathcal{T} = \{\phi, X\}$

Then for any point $p \in X$, we have X is the only open set which contains p, and hence

$(X-\{P\}) \cap A \neq \emptyset$, where $A \subset X$

$\therefore$ P is a limit Point of A, if $A \neq \emptyset$

The derived set of A, in general is:

$$A' = \begin{cases} \emptyset, & \text{if } A = \emptyset \\ \{P\}' = X - \{P\}, & A = \{P\} \\ X, & \text{if A contains two elements or more} \end{cases}$$

Closure of a Set:

Let A be a subset of a topological space X, the closure of A (denoted by $\bar{A}$) is the intersection of all closed sets containing A, i.e., $\bar{A} = \bigcap_{i \in I} F_i$, where $F_i, \forall i \in I$ is a closed set and $A \subseteq F_i, \forall i$.

Remarks:

1- $\bar{A}$ is closed set, since the intersection of closed sets is closed.

2- $\bar{A}$ is the smallest closed set containing A, i.e., if F is any other closed set such that $A \subset F$, then $A \subseteq \bar{A} \subseteq F$.

3- The set A is closed iff $A = \bar{A}$.

Theorem:

Let (X, τ) be a topological space and $A, B \subseteq X$, such that $A \subset B$, then $A' \subset B'$.

Proof:

Let $P \in A' \rightarrow \forall$ open set G, such that $P \in G$, we have $(G-\{P\}) \cap A \neq \emptyset$

$\because A \subset B$

$\therefore \emptyset \neq (G-\{P\}) \cap A \subset (G-\{P\}) \cap B$

$\rightarrow (G-\{P\}) \cap B \neq \emptyset \rightarrow P$ is a limit Point of B

$\rightarrow P \in B'$

$\therefore A' \subset B'$ #

Theorem:

Let A and B be two subsets of a topological space (X,τ). Then $(A\cup B)' = A'\cup B'$

Proof:

$\because A \subseteq A\cup B \rightarrow A' \subseteq (A\cup B)'$

$B \subseteq A\cup B \rightarrow B' \subseteq (A\cup B)'$

$\therefore A'\cup B' \subseteq (A\cup B)'$ --- (1)

T.P. $(A\cup B)' \subseteq A'\cup B'$

Let $p \in (A\cup B)' \rightarrow \exists$ an open set G such that $\emptyset \neq (G-\{p\})\cap(A\cup B)$

$= [(G-\{p\})\cap A]\cup[(G-\{p\})\cap B]$

clear that either $(G-\{p\})\cap A \neq \emptyset$ or $(G-\{p\})\cap B \neq \emptyset$ or both (since their union is nonempty)

If $(G-\{p\})\cap A \neq \emptyset \rightarrow p$ is a limit point of A

$\rightarrow p \in A' \rightarrow p \in A'\cup B'$ (even so if B' is empty)

Similarly, If $(G-\{p\})\cap B \neq \emptyset \rightarrow p \in B'$ and hence $p \in A'\cup B'$

$\therefore (A\cup B') \subseteq A'\cup B'$ --- (2)

From (1) and (2), we have

$(A\cup B)' = A'\cup B'$ #

Example:

Let $X = \{a,b,c,d,e\}$ and define a topology on X by

$\tau = \{\emptyset, X, \{a\}, \{a,b\}, \{a,c,d\}, \{a,b,c,d\}, \{a,b,e\}\}$

1- Find the closed subsets of X

2- Determine the closure of the sets $\{a\}, \{b\}, \{c,e\}$.

Solution:

1- The open subsets of X are

$\emptyset, X, \{a\}, \{a,b\}, \{a,c,d\}, \{a,b,c,d\}, \{a,b,e\}$

Hence, the closed sets in X are the complement of the open sets, i.e., the closed sets are

$X, \emptyset, \{b,c,d,e\}, \{c,d,e\}, \{b,e\}, \{e\}, \{c,d\}$

2- If $A=\{a\}$, then the closed sets containing A is only X and hence $\bar{A}=\overline{\{a\}}=X$

If $A=\{b\}$, then the closed sets containing A are $F_1=X$, $F_2=\{b,c,d,e\}$, $F_3=\{b,e\}$,

$\Rightarrow \bigcap_{i=1}^{3} F_i=\{b,e\}=\bar{A}=\overline{\{b\}}$

Similarly, if $A=\{c,e\}$, then the closed sets containing A are $F_1=X$, $F_2=\{b,c,d,e\}$, $F_3=\{c,d,e\}$

$\Rightarrow \bigcap_{i=1}^{3} F_i=\{c,d,e\}=\bar{A}=\overline{\{c,e\}}$

H.W:

① If F is closed subset of a topological space (X,τ) and $A\subset F$, then $A'\subset F$.

② If $A\subset B$, then $\bar{A}\subset\bar{B}$.

Theorem:

If A and B are two subsets of a topological space (X,τ), then $\overline{A\cup B}=\bar{A}\cup\bar{B}$.

Proof:

Since $A\subseteq A\cup B \rightarrow \bar{A}\subseteq\overline{A\cup B}$

$B\subseteq A\cup B \rightarrow \bar{B}\subseteq\overline{A\cup B}$

$\Rightarrow \bar{A}\cup\bar{B}\subseteq\overline{A\cup B}$ --- ①

Also $A\subseteq\bar{A}$, $B\subseteq\bar{B}\rightarrow A\cup B\subseteq\bar{A}\cup\bar{B}$ which is a closed set, since the union of closed sets is closed

$\Rightarrow A\cup B\subseteq\overline{A\cup B}\subseteq(\bar{A}\cup\bar{B})=F$ closed (see page)

$\rightarrow \overline{A\cup B}\subseteq\bar{A}\cup\bar{B}$ --- ②

From ① and ②, we have

$$\overline{A\cup B}=\bar{A}\cup\bar{B} \quad \#$$

Proposition: If (X,τ) is a topological space, then (1) $\bar{\emptyset}=\emptyset$, (2) $\overline{(\bar{A})}=\bar{A}$.

Proof:

Since $\emptyset$ and $\bar{A}$ are closed sets (by definition) and by the remark which says that if a set

F is closed then $F = \bar{F}$.
Hence $\emptyset = \bar{\emptyset}$ and $(\bar{A})' = \bar{A}$ #

Interior, Exterior and Boundary Points:

Definition:
Let A be a subset of a topological space (X, τ), then a point $p \in A$ is called an interior point of A if there exist an open set G, such that $p \in G \subset A$.

The set of all interior points of A is denoted by $\text{Int}(A)$ or A°.

Example:
Let $X = \{a, b, c, d, e\}$ and define a topology on X by $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, c, d\}, \{a, b, e\}\}$
Let $A = \{a, b, c\}$ and to find A°.

Solution:
$a \in A$ and the open sets containing a are $\{a\}, \{a, b\}$ $\{a, c, d\}, \{a, b, c, d\}, \{a, b, e\}$ and since $a \in \{a\} \subset A = \{a, b, c\}$
i.e., $\exists$ an open set $G = \{a\}$ such that $a \in G \subset A$
$\therefore$ a is an interior point
Similarly, $b \in A$ and there exist an open set $G = \{a, b\}$ such that $b \in G \subset A$
$\therefore$ b is an interior point of A
But c is not an interior point since the open sets containing c are $X, \{a, c, d\}, \{a, b, c, d\}$ and non of them is a subset of A
$\therefore A^\circ = \{a, b\}$

Definitions:
① The exterior point of a subset A of a topological space (X, T) is that point which is interior point of A^c.

(2) The boundary Point of a subset A of a topological space (X,T) is that Point which is not an interior nor exterior Point of A.

The set of all exterior Points will be denoted by $Ext(A)$ and the set of all boundary Points will be denoted by $bd(A)$.

Remarks:

$A^\circ = \bigcup_{i\in I} G_i$, where G_i is an open subset of A and furthermore

1- A° is open

2- A° is the largest open subset of A, i-e., If G is open subset of A, then $G \subset A^\circ \subset A$.

3- A is open iff $A = A^\circ$.

Proof:

Let $\{G_i\}_{i\in I}$ be a collection of open subsets of A, $G_i \subset A$. If $x \in A^\circ \rightarrow$ ∃ an open set G_{i_0} such that

$$x \in G_{i_0} \subset A$$

$$\rightarrow x \in \bigcup_{i\in I} G_i \rightarrow A^\circ \subset \bigcup_{i\in I} G_i \quad \text{---(1)}$$

Now, let $x \in \bigcup_{i\in I} G_i \rightarrow x \in G_{i_0}$, for some i_0

$\rightarrow$ ∃ at least one open set G_{i^*} such that

$$x \in G_{i^*} \subset A \rightarrow x \in A^\circ \rightarrow \bigcup_{i\in I} G_i \subset A^\circ \quad \text{---(2)}$$

From (1) and (2), we have

$$A^\circ = \bigcup_{i\in I} G_i,\ G_i \subset A,\ G_i \text{ is open}, \forall i$$

Also:

1- $A^\circ = \bigcup_{i\in I} G_i$ is open set (from the second condition of topological spaces) $\rightarrow A^\circ$ is open

2. If G is an open subset of A $\rightarrow G \in \{G_i\}_{i\in I}$

$$\rightarrow G \subset \bigcup_{i\in I} G_i = A^\circ \subset A$$

i.e., A° is the largest open set contained in A

3- If A is open $\rightarrow A \subset A^\circ \subset A$ or $A = A^\circ$ and when $A = A^\circ$ (open), then A is open. #

Theorem:

Let (X, τ) be a topological space and $A, B \subseteq X$, then

(1) If $A \subseteq B$, then $A^\circ \subseteq B^\circ$.

(2) $A^\circ \cap B^\circ = (A \cap B)^\circ$

(3) $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$

Proof:

(1) Let $A \subseteq B$ and if $x \in A^\circ$, then by definition there exist an open set G, such that $x \in G \subset A$

$\therefore A \subseteq B \rightarrow x \in G \subset B$ (G is open set)

$\rightarrow x \in B^\circ$

$\therefore A^\circ \subseteq B^\circ$

(2) T.P. $A^\circ \cap B^\circ = (A \cap B)^\circ$, i.e., T.P. $(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$ and $A^\circ \cap B^\circ \subseteq (A \cap B)^\circ$

$\because A \cap B \subseteq A$ and $A \cap B \subseteq B$ (by the def. of $\cap$)

$\rightarrow$ by (1), we have $(A \cap B)^\circ \subseteq A^\circ \wedge (A \cap B)^\circ \subseteq B^\circ$

$\rightarrow (A \cap B)^\circ \subseteq A^\circ \cap B^\circ$ ---(i)

Now, Since $A^\circ \subseteq A \wedge B^\circ \subseteq B$

$\rightarrow A^\circ \cap B^\circ \subseteq A \cap B$

$\therefore A^\circ \cap B^\circ$ is an open subset of $A \cap B$ and $(A \cap B)^\circ$ is the largest subset of $A \cap B$

$\rightarrow A^\circ \cap B^\circ \subseteq (A \cap B)^\circ \subseteq A \cap B$

$\rightarrow A^\circ \cap B^\circ \subseteq (A \cap B)^\circ$ ---(ii)

From (i) and (ii), we have $(A \cap B)^\circ = A^\circ \cap B^\circ$.

(3) $\because A \subseteq A \cup B \wedge B \subseteq A \cup B$ (by def. of the union)

$\rightarrow$ by (1), we have

$A^\circ \subseteq (A \cup B)^\circ \wedge B^\circ \subseteq (A \cup B)^\circ$

$\rightarrow A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$

The converse is not true, as it can be seen in the following counter example:

Example:

Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}\}$

and suppose that $A = \{a\}$, $B = \{b, c\}$

Then $A^\circ = \emptyset$, $B^\circ = \{b, c\} = B$

$\therefore A^\circ \cup B^\circ = \phi \cup B = \{b,c\}$

Now, $A \cup B = \{a\} \cup \{b,c\} = \{a,b,c\} = X$ and it is clear that $(A \cup B)^\circ = X$, i.e., $(A \cup B)^\circ \neq A^\circ \cup B^\circ$ since $(A \cup B)^\circ = \{a,b,c\} = X \neq \{b,c\} = A^\circ \cup B^\circ$ #

Theorem:

Let (X, T) be a topological space and $A, B \subseteq X$, then

1- $Int(A) \cap Ext(A) = \phi$.

2- If $A \subseteq B$, then $Ext(B) \subseteq Ext(A)$.

3- $Ext(A \cup B) = Ext(A) \cap Ext(B)$.

4- $A^c \in T$, i.e., A is closed iff $Ext(A) = A^c$.

5- $Ext(A) \cup Ext(B) \subseteq Ext(A \cap B)$.

Proofs

1- From the definition of $Int(A) \rightarrow Int(A) \subseteq A \wedge Ext(A) \subseteq A^c$

$\rightarrow Int(A) \cap Ext(A) \subseteq A \cap A^c = \phi$

$\rightarrow Int(A) \cap Ext(A) \subseteq \phi$

$\rightarrow Int(A) \cap Ext(A) = \phi$

2- Suppose that $A \subseteq B$, T.P. $Ext(B) \subseteq Ext(A)$

Let $x \in Ext(B) \rightarrow \exists G \in T$, such that $x \in G \subset B^c$ (by def.)

$\rightarrow \exists G \in T$, s.t $x \in G \subset A^c$ ($A \subseteq B \rightarrow B^c \subseteq A^c$)

i.e., x is an interior point of A^c

$\rightarrow x \in Ext(A)$

$\therefore Ext(B) \subseteq Ext(A)$.

3- $Ext(A \cup B) = Int[(A \cup B)^c] = Int(A^c \cap B^c)$

$= Int(A^c) \cap Int(B^c)$

$= Ext(A) \cap Ext(B)$.

4- $\Rightarrow$) If A is closed $\rightarrow A^c$ is open, T.P $Ext(A) = A^c$

$\because A^c \in T \rightarrow Int(A^c) = A^c$ (A is open iff $A = A^\circ$)

$\rightarrow Ext(A) = A^c$

$\Leftarrow$) If $Ext(A) = A^c$, T.P. A is closed or A^c is open

$\because Ext(A) = A^c \rightarrow Int(A^c) = A^c$

$\rightarrow A^c$ is open (A is open iff $A = A^\circ$)

$\rightarrow A$ is closed

3- ∵ $A \cap B \subseteq A \wedge A \cap B \subseteq B$

$\rightarrow Ext(A) \subseteq Ext(A \cap B) \wedge Ext(B) \subseteq Ext(A \cap B)$

$\rightarrow Ext(A) \cup Ext(B) \subseteq Ext(A \cap B)$. #

Remark:

Recall the boundary point which is neither interior nor exterior which is equivalent to say that a point $x \in X$ is said to be boundary point of a set $A \subseteq X$ iff every open set in X containing x contains at least one point of A and at least one point of A^c, in which the set of all boundary points of A is denoted by $bd(A)$, where

$$bd(A) = \{x \in X : \forall G \in T, x \in G, G \cap A \neq \emptyset \wedge G \cap A^c \neq \emptyset\}$$

Theorem:

Let (X, T) be a topological space and $A, B \subseteq X$, then:

1- $bd(A) \cap Int(A) = \emptyset$, $bd(A) \cap Ext(A) = \emptyset$.

2- $bd(A) = bd(A^c)$.

3- $bd(A \cup B) \subseteq bd(A) \cup bd(B)$.

4- $A \in T$, i.e., A is open iff $bd(A) \subseteq A^c$ and $bd(A) \cap A = \emptyset$.

5- $A^c \in T$, i.e., A^c is open iff $bd(A) \subseteq A$ and $bd(A) \cap A^c = \emptyset$.

6- $A, A^c \in T$, i.e., A and A^c are open iff $bd(A) = \emptyset$

Proof:

1- T.P. $bd(A) \cap Int(A) = \emptyset$, let $bd(A) \cap Int(A) \neq \emptyset$

$\rightarrow \exists x \in bd(A)$ and $x \in Int(A)$

∵ $x \in Int(A) \rightarrow \exists G \in T$, such that $x \in G \subseteq A$ (def.)

$\rightarrow G \cap A^c = \emptyset$ (since $G \subseteq A$)

$\rightarrow x \notin bd(A)$ (by the above remark)

which is a contradiction

∴ $bd(A) \cap Int(A) = \emptyset$

Similarly, to prove $bd(A) \cap Ext(A) = \emptyset$

Let $bd(A) \cap Ext(A) \neq \emptyset$

$\rightarrow \exists x \in bd(A)$ and $x \in Ext(A)$

∵ $x \in Ext(A) \rightarrow \exists G \in T$, s.t. $x \in G \subseteq A^c$ (by def. of $bd(A)$)

$\rightarrow G \cap A = \emptyset$ (since $G \subseteq A^c$)

$\rightarrow x \notin bd(A)$ (by the last remark)

which is a contradiction

$\therefore bd(A) \cap Ext(A) = \emptyset$

2- From the definition of $bd(A)$, we have:

$x \in bd(A) \leftrightarrow \forall G \in T$, such that $G \cap A \neq \emptyset$ and $G \cap A^c \neq \emptyset$

$\leftrightarrow \forall G \in T, x \in G$, $G \cap (A^c)^c \neq \emptyset$ and $G \cap A^c \neq \emptyset$ (since $(A^c)^c = A$)

$\leftrightarrow x \in bd(A^c)$

$\therefore bd(A) = bd(A^c)$

3- T.P. $bd(A \cup B) \subseteq bd(A) \cup bd(B)$

Let $x \in bd(A \cup B) \rightarrow \forall G \in T, x \in G$, such that

$G \cap (A \cup B) \neq \emptyset$ and $G \cap (A \cup B)^c \neq \emptyset$

$\rightarrow [(G \cap A) \cup (G \cap B)] \neq \emptyset$ and $G \cap (A^c \cap B^c) \neq \emptyset$

$\rightarrow [(G \cap A) \cup (G \cap B)] \neq \emptyset$ and $[(G \cap A^c) \cap (G \cap B^c)] \neq \emptyset$

$\rightarrow [G \cap A \neq \emptyset \vee G \cap B \neq \emptyset] \wedge [G \cap A^c \neq \emptyset \wedge G \cap B^c \neq \emptyset]$

$\rightarrow [G \cap A \neq \emptyset \wedge G \cap A^c \neq \emptyset] \vee [G \cap B \neq \emptyset \wedge G \cap B^c \neq \emptyset]$

$\rightarrow x \in bd(A) \vee x \in bd(B)$

$\rightarrow x \in bd(A) \cup bd(B)$

$\therefore bd(A \cup B) \subseteq bd(A) \cup bd(B)$

4- $\Rightarrow$ Suppose that $A \in T$, T.P. $bd(A) \subseteq A^c$ and $bd(A) \cap A = \emptyset$

Let $x \in bd(A) \rightarrow \forall G \in T, x \in G$, $G \cap A \neq \emptyset \wedge G \cap A^c \neq \emptyset$

i.e., every open set G containing x intersects A & A^c

Since A is open and $A \cap A^c = \emptyset \rightarrow x \notin A$

$\rightarrow x \in A^c \rightarrow bd(A) \subseteq A^c$

$\Leftarrow$ If $bd(A) \subseteq A^c$, T.P. A is an open set

i.e., T.P. $A \in T$, i.e., T.P. A contains an open set G for every point of A, s.t. $x \in G \subseteq A$

Let $x \in A \rightarrow x \notin A^c \rightarrow x \notin bd(A)$ (since $bd(A) \subseteq A^c$)

$\rightarrow \exists G \in T, x \in G$ s.t. $G \cap A = \emptyset \vee G \cap A^c = \emptyset$ (is obvious)

But $G \cap A \neq \emptyset$, since $x \in A$ and $x \in G$

$\therefore G \cap A^c = \emptyset$

$\therefore G \subseteq A \rightarrow x \in G \subseteq A \rightarrow A$ is open set

5- $\Rightarrow$ If $A^c \in T$, T.P. $bd(A) \subseteq A$ and $bd(A) \cap A^c = \emptyset$

Let $x \in bd(A) \rightarrow \forall$ open set $G \in T$, $x \in G$, $G \cap A \neq \emptyset$ $\wedge$ $G \cap A^c \neq \emptyset$ (by def. of bd)

$\because A^c$ is open $\rightarrow x \notin A^c$ (G)

Now, since every open set containing x intersect A and A^c, then x cannot be in A^c (since A^c must contain the open neighborhood for every point of A^c).

$\rightarrow x \in A$, because $X = A \cup A^c$

$\rightarrow bd(A) \subseteq A$ and also implise $bd(A) \cap A^c = \emptyset$

$\Leftarrow$ Suppose that $bd(A) \subseteq A$, T.P. A is closed (i.e., T.P $A^c \in T$, i.e., A^c is open)

In order to prove that A^c is open, we must prove that A^c contain open neighborhood (openset) for every point in A^c

Let $x \in A^c \rightarrow x \notin A \rightarrow x \notin bd(A)$, since $bd(A) \subseteq A$

$\rightarrow \exists$ an open set $G \in T$, $x \in G$, such that $G \cap A = \emptyset$ (by the def. of boundary points and since $x \in A^c$)

$\rightarrow G \subseteq A^c \neq \emptyset$ (because $X = A \cup A^c$)

$\therefore A^c$ contains an open set (neighborhood) for every point in A^c

$\therefore A^c \in T$, i.e., A^c is open and hence also we have A is closed.

6- $\Rightarrow$ Suppose that $A, A^c \in T$, T.P. $bd(A) = \emptyset$

$\because A$ is open $\xrightarrow[(5)]{\text{by}} bd(A^c) \subseteq A^c \rightarrow bd(A) \subseteq A^c$ (since $bd(A) = bd(A^c)$)

$\because A$ is closed $\rightarrow bd(A) \subseteq A$

$\therefore bd(A) \subseteq A \cap A^c = \emptyset \rightarrow bd(A) = \emptyset$

$\Leftarrow$ If $bd(A) = \emptyset$, T.P. $A, A^c \in T$

$\because bd(A) = \emptyset$ and $\emptyset \subseteq A$ and $\emptyset \subseteq A^c$

$\therefore bd(A) \subseteq A \rightarrow A^c$ is open, i.e., $A^c \in T$

$bd(A) \subseteq A^c \rightarrow A$ is open, i.e., $A \in T$

#

Theorem:

Let (X,τ) be a topological space and $A \subset X$, then $A \cup A'$ is a closed set.

Proof: سنبرهن أن المتممة مفتوحة

Let $x \in (A \cup A')^c \rightarrow x \notin A \cup A'$

$\rightarrow x \notin A \wedge x \notin A'$

Now, $x \notin A' \rightarrow \exists$ an open set G, such that

$x \in G$, $G \cap A = \phi$ or $\{p\}$ اي ان التقاطع لا يمكن ان يكون نقطة

Since $x \notin A$, then in particular $G \cap A = \phi$ ... ①

Also, we can prove that $G \cap A' = \phi$, since if $p \in G$, then $g \in G$ and $G \cap A = \phi$, where G is an open set

$\therefore g \notin A' \rightarrow G \cap A' = \phi$ ... ②

$\therefore G \cap (A \cup A') = G \cap A) \cup (G \cap A')^c$

$\xrightarrow[①②]{by} \phi \cup \phi = \phi$

$\rightarrow G \subset (A \cup A')^c$

$\therefore x$ is an interior point of $(A \cup A')^c$

$\therefore (A \cup A')^c$ is open

$\rightarrow A \cup A'$ is closed #

Theorem:

Let (X,τ) be a topological space and $A \subset X$, then $\bar{A} = A \cup A'$

Proof:

Since $A \subseteq \bar{A}$ and $\bar{A}$ is closed

$\rightarrow A' \subseteq (\bar{A})' = \bar{A}$ $\quad \bar{A} \cup \bar{A} = \bar{A}$

$\rightarrow A \cup A' \subseteq \bar{A}$ ... ①

$\because A \cup A'$ is a closed set, which contains A

$\rightarrow A \subseteq \bar{A} \subseteq A \cup A'$ لانه $\bar{A}$ اصغر مجموعة مغلقة تحوي A

$\rightarrow \bar{A} \subseteq A \cup A'$ ... ②

from ① and ②, we have $\bar{A} = A \cup A'$. #

Definition:

Let (X, τ) be a topological space and A be a subset of X (i.e., $A \subset X$). Then

$$\tau_A = \{A \cap G : G \in \tau\}$$

is a topology on A, which is called the relative topology on A or the relativization of τ on A. Also, (A, τ_A) is a subspace of (X, τ).

In other words, if $H \subset A$ is a τ_A-open set, i.e., open relative to A iff there exist $G \subset X$ is τ-open set, such that $H = G \cap A$.

Example:

Consider $X = \{a, b, c, d, e\}$ and consider a topology on X to be $\tau = \{\emptyset, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$

Let $A = \{a, d, e\} \subset X$, then

$X \cap A = A$, $\emptyset \cap A = \emptyset$, $\{a\} \cap A = \{a\}$, $\{c, d\} \cap A = \{d\}$

$\{a, c, d\} \cap A = \{a, d\}$ and $\{b, c, d, e\} \cap A = \{d, e\}$

Hence, the relative topology on A is

$$\tau_A = \{\emptyset, A, \{a\}, \{d\}, \{a, d\}, \{d, e\}\}$$

Therefore, the open sets relative to A are

$\emptyset$, A, $\{a\}$, $\{d\}$, $\{a, d\}$, $\{d, e\}$.

Bases and Subbases

Let (X, τ) be a topological space and $\mathcal{B}$ be a subfamily from τ (i.e., class of open subsets of X), then $\mathcal{B}$ is a basis or base of the topology τ if every element of τ (open set) is a union of members of $\mathcal{B}$, i.e., $\mathcal{B}$ is a basis or base for τ iff:

$$1- \mathcal{B} \subseteq \tau \quad , \quad 2- \forall G \in \tau, G = \bigcup_{i \in I} B_i, B_i \in \mathcal{B}, \forall i$$

Remark:

From the definition of the base, it is noted that the number of bases are not determined, may be finite or may be infinite.

Example:

Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Find a base for τ.

Solution:

Let $\mathcal{B} = \{X, \emptyset, \{a\}, \{b\}\}$

It is clear that $\mathcal{B} \subseteq \tau$ and the members of $\mathcal{B}$ are $X, \emptyset, \{a\}, \{b\} \in \tau$ and also belong to $\mathcal{B}$, and $\{a, b\} \in \tau \longrightarrow \{a, b\} = \{a\} \cup \{b\}$

$\therefore \mathcal{B}$ is a basis for τ.

Remark:

The topology τ is a base for τ (by choosing $\mathcal{B} = \tau$) and this base is called the trivial basis or base and so every topology has a basis (which is at least the trivial base).

Example:

Let $X = \{1, 2, 3\}$ and $\tau_{ind.} = \{\emptyset, X, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$. Define two different basis on τ.

Solution

Let $\mathcal{B}_1 = \{\emptyset, \{1\}, \{2\}, \{3\}\}$, $\mathcal{B}_2 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}\}$

we can show that $\mathcal{B}_1$ and $\mathcal{B}_2$ are bases for τ since every one of them generates the elements of τ.

Q: (H.W.)

Let (X, T) be a topological subspace of (Y, T^*) and let (Y, T^*) be also a topological subspace of (Z, T^{**}). Show that (X, T) is also a topological subspace of (Z, T^{**}).

T.P. W is a subbasis for for T, we must find all possible intersections of elements of W in order to get $\mathcal{B}$. Now:

$\emptyset = \{a,c\} \cap \{b\} \in \mathcal{B}$, $\{a,c\} \cap \{a,b\} = \{a\} \in \mathcal{B}$,
$\{b\} \cap \{b\} = \{b\} \in \mathcal{B}$, $\{a,c\} \cap \{a,c\} = \{a,c\} \in \mathcal{B}$,
$\{a,b\} \cap \{a,b\} = \{a,b\} \in \mathcal{B}$, $X \cap X = X \in \mathcal{B}$

∴ W is a subbasis of T.

Example:

Define a subbasis for the usual topological space $(\mathbb{R}, \tau_u)$.

Solution:

Recall that we had proved that $\mathcal{B} = \{(a,b) : a,b \in \mathbb{R}\}$ is a basis for the usual topology τ_u. To define a subbasis W for τ_u, such that W generated $\mathcal{B}$

Define $W = \{(a,b) : a = -\infty \text{ or } b = \infty\}$

T.P. W is a subbasis for τ_u, take $H \in \mathcal{B}$ and to prove that it is equal to a finite intersection of elements of W

Let $(a,b) \in \mathcal{B}$, $a,b \in \mathbb{R}$, s.t. $b > a$

$\rightarrow (a,b) = (-\infty, b) \cap (a, \infty)$
$\in W \quad \in W$



∴ W is a subbasis of τ_u

Remarks:

1- We can find more than one subbasis for a certain topology
2- Every subbasis for any topology must contains X.
3- $\emptyset$ may not belong to the subbasis.
4- The topology T is a subbasis for T (by choosing W=T) and this subbasis is called also the trivial subbasis

Question:

Show that a basis (subbasis) determines one and only one topology, (i.e., the topology generated by the basis or subbasis is unique).

Solution:

Suppose that $\mathcal{B}$ is a basis for two topologies T and T^* and to prove that $T = T^*$, i.e., T.P. $T \subseteq T^*$ and $T^* \subseteq T$

Let $G \in T \rightarrow G = \bigcup_{i \in I} B_i$, $\forall B_i \in \mathcal{B} \subseteq T^*$ and since T^* is a topology $\rightarrow \bigcup_{i \in I} B_i \in T^*$, i.e., $G \in T^* \rightarrow T \subseteq T^*$

Also, $G \in T^*$, then $G = \bigcup_{i \in I} B_i$, $\forall B_i \in \mathcal{B} \subseteq T$

$\rightarrow G \in T \rightarrow T^* \subseteq T$

$\therefore T = T^*$

Now, $W = \{W_i\}_{i \in I}$ is a subbasis for two topologies T and T^* on X and T.P. $T = T^*$

Let $G \in T \rightarrow G = \bigcup_{i \in I} (W_{i_1} \cap W_{i_2} \cap \dots \cap W_{i_{n_i}})$, $\forall W_{i_k} \in W \subseteq T^*$ and since T^* is a topology, then the intersection of infinite collection of T^* and arbitrary union of collection of sets of T^* belongs to $T^* \rightarrow G \in T^* \rightarrow T \subseteq T^*$

Similarly, we can prove $T^* \subseteq T$

$\therefore T = T^*$ ##

Property:

Let (X, T) be a topological space, $\mathcal{B}$ be a basis of T and $\mathcal{B}^*$ be a family of open sets, such that $\mathcal{B} \subseteq \mathcal{B}^* \subseteq T$. Then $\mathcal{B}^*$ is also a basis of T.

Proof:

$\because \mathcal{B}$ is a basis of T

$\therefore \mathcal{B} \subseteq T$ and if $G \in T$, then $G = \bigcup_{i \in I} B_i$, $B_i \in \mathcal{B}$

T.P. $\mathcal{B}^*$ is a basis for T

Clear that $\mathcal{B}^* \subseteq T$ and $\mathcal{B} \subseteq \mathcal{B}^*$, so $B_i \in \mathcal{B} \subseteq \mathcal{B}^* \rightarrow B_i \in \mathcal{B}^*, \forall$

$\therefore G = \bigcup_{i \in I} B_i$, $B_i \in \mathcal{B}$

$= \bigcup_{i \in I} B_i$, $B_i \in \mathcal{B}^*$

$\therefore \mathcal{B}^*$ a basis for T ##

Definition:

We say that $\mathcal{B}$ and $\mathcal{B}^*$ are equivalent basis if they yield (give) the same topology.

Remarks:

Two basis $\mathcal{B}$ and $\mathcal{B}^*$ of X are equivalent iff

1- For all $B_i \in \mathcal{B}$ and $x \in \mathcal{B}$, there exist $B_i^* \in \mathcal{B}^*$ with $x \in B_i^* \subset B$.

2 For all $B_i^* \in \mathcal{B}^*$ and $x \in B^*$, there exist $B_i \in \mathcal{B}$ with $x \in B \subset B^*$.

Proof:

$\Rightarrow$ If $\mathcal{B}$ and $\mathcal{B}^*$ are an equivalent basis of X, then $\mathcal{B}$ and $\mathcal{B}^*$ yields the same topology on X

Let $B_i \in \mathcal{B}$ and $x \in B \rightarrow x \in B \in T$. But $\mathcal{B}^*$ is a basis for T, then there exists $B_i^* \in \mathcal{B}^*$ with $x \in B_i^* \subset B$ and hence (1) is hold. Similarly, we can show 2.

$\Leftarrow$ Suppose that Conditions (1) and (2) are satisfied

$\therefore$ $\mathcal{B}$ yield the topology $T_{\mathcal{B}}$ and $\mathcal{B}^*$ yields the topology $T_{\mathcal{B}^*}$ and T.P. $T_{\mathcal{B}} = T_{\mathcal{B}^*}$

Let $G \in T_{\mathcal{B}} \rightarrow G = \bigcup_{i \in I} B_i, B_i \in \mathcal{B}$ and if $x \in G$, then there exists $i_0 \in I$, such that $x \in B_{i_0}$. By (1), there exist $B_i^* \in \mathcal{B}^*$ with $x \in B_i^* \subseteq B_{i_0} \subseteq G$

$\therefore$ $G \in T_{\mathcal{B}^*} \rightarrow T_{\mathcal{B}} \subseteq T_{\mathcal{B}^*}$

Similarly, we can prove $T_{\mathcal{B}^*} \subseteq T_{\mathcal{B}}$

$\therefore$ $T_{\mathcal{B}} = T_{\mathcal{B}^*}$

Hence the two basis which generates the same topol are equivalent. #

Theorem:

Given any family $S = \{S_i\}_{i \in I}$ of subsets of X such that $X = \bigcup_{i \in I} S_i$ (unnecessary). Then the topology generated by S is characterized as follows:

$T(S) = \{\emptyset, X$, all finite intersections of S_i, and all arbitrary unions of these finite intersections$\}$

and S is a subbasis for $T(S)$.

Proof:

Let

$T^* = \{\emptyset, X$, all finite intersections of S_i, and all arbitrary unions of these finite intersections$\}$

$= \{$all unions of finite intersections of $S_i\}$

$\mathcal{B}(S) = \{$all finite intersections of the $S_i\} = \{\bigcap_{i=1}^{n} S_i : S_i \in S\}$

Since $\mathcal{B}(S)$ is a basis for some topology on X and this basis determines only one topology, then T^* is a topology and it is the topology with basis $\mathcal{B}(S)$ and subbasis S.

Then the proof is completed if we show that $T^* = T(S)$

Since $T(S)$ is a topology, then $T^* \subseteq T(S)$ (by the definition of T^*).

Also, since T^* is a topology containing S, then

$T(S) \subseteq T^*$ (by definition of $T(S)$)

$\Rightarrow T^* = T(S)$. #